AN EXPOSITORY NOTE ON THE WELL-POSEDNESS OF THE TRANSPORT EQUATION UNDER STOCHASTIC PERTURBATIONS

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Abstract. This note provides a summary and commentary on the work of Flandoli, Gubinelli, and Priola (2010) in [1] on the well-posedness of the transport equation under stochastic perturbations. The original paper establishes existence, uniqueness, and stability results for the transport equation driven by stochastic noise, using tools from stochastic analysis and PDE theory. Here, we aim to present the main results, outline key techniques, and offer some insights and perspectives for readers seeking a simpler understanding of these developments. All results and proofs discussed are due to the original authors; this note serves solely as an exposition and commentary.

Contents

1. Introduction

Definition 1. *Let $b \in \mathbb{L}^1_{loc}([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$, $\operatorname{div} b \in \mathbb{L}^1_{loc}([0,T];\mathbb{R}^d)$ and $u_0 \in \mathbb{L}^\infty(\mathbb{R}^d)$. We call $u \in \mathbb{L}^\infty(\Omega\times[0,T]\times\mathbb{R}^d)$ a solution if for every $\theta \in C_c^\infty(\mathbb{R}^d)$ the process $\int_{\mathbb{R}^d}\theta(x)u(t,x)dx$ has a continuous modification which is an $\mathcal{F}_t$-semimartingale and:*

$$\begin{aligned}\int_{\mathbb{R}^d} u(t,x)\theta(x)dx &= \int_{\mathbb{R}^d} u_0(x)\theta(x)dx + \int_0^t ds\int_{\mathbb{R}^d} u(s,x)\left[b(s,x)\cdot\nabla\theta(x) + \operatorname{div} b(s,x)\theta(x)\right]dx \\ &\quad + \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u(s,x)\nabla_i\theta(x)dx\right)\circ dW_s^i.\end{aligned} \tag{1}$$

$$\begin{cases} dX_t(x) = b(t,X_t(x))dt + dW_t, \\ X_0(x) = x, \quad x\in\mathbb{R}^d. \end{cases} \tag{2}$$

2. Existence

3. Regularity of the Jacobian

4. Uniqueness

The uniqueness argument uses an estimate on a commutator as in the deterministic theory. Let us start with this classical result:

Lemma 1. *Let ϕ a C^1 diffeomorphism of $\mathbb{R}^d$. Assume $v \in \mathbb{L}^\infty_{loc}(\mathbb{R}^d,\mathbb{R}^d)$, $\operatorname{div} v \in \mathbb{L}^1_{loc}(\mathbb{R}^d)$ and $g \in \mathbb{L}^1_{loc}$. Assume for every $\rho \in C_c^\infty(\mathbb{R}^d)$ there exists C_ρ such that for every $R>0$:*

$$\left|\int_{\mathbb{R}^d}\mathcal{R}_\epsilon[v,g](\phi(x))\rho(x)dx\right| \le C_\rho \|g\|_{\mathbb{L}^\infty_{R+1}}\left[\|\operatorname{div} v\|_{\mathbb{L}^1_{R+1}}\left\|J\phi^{-1}\right\|_{\mathbb{L}^\infty_R} + \|v\|_{\mathbb{L}^\infty_{R+1}}\left(\left\|D\phi^{-1}\right\|_{\mathbb{L}^\infty_R} + \left\|DJ\phi^{-1}\right\|_{\mathbb{L}^1_R}\right)\right].$$

Then the following limit holds:

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\mathcal{R}_\varepsilon[v,g](\phi(x))\rho(x)dx = 0.$$

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Proof. Just notice:

$$\int_{\mathbb{R}^d} \mathcal{R}_\varepsilon[v,g](\phi(x))\rho(x)dx = \int_{\mathbb{R}^d} \mathcal{R}_\varepsilon[v,g](y)\rho_\phi(y)dy,$$

where:

$$\rho_\phi(y) = \rho(\phi^{-1}(y))J\phi^{-1}(y).$$

Now the proof continues as in Chapter 2 in [3]. □

We are now ready to prove the following result:

Theorem 2. *Assume $b \in \mathbb{L}^\infty([0,T]; C_b^\alpha(\mathbb{R}^d;\mathbb{R}^d))$ for $\alpha \in (0,1)$, and assume that for $p > 2$ it holds $\operatorname{div} b \in \mathbb{L}^p([0,T]\times\mathbb{R}^d)$. Then there exists a unique $\mathbb{L}^\infty$ weak solution in the sense of Definition 1.*

Proof. By linearity, it is enough to prove that if $u_0 = 0$ and u is a solution in the sense of Definition 1, then $u(t,\cdot) \equiv 0$ for all $t \in [0,T]$, and choose $\theta_\varepsilon(x-\cdot)$ in Definition 1. Defining $u_\varepsilon(t,x) := \theta_\varepsilon \star u(t,\cdot)(x)$, one has:

$$u^\varepsilon(t,y) = \int_0^t A_\varepsilon(s,y)ds + \sum_{i=1}^d \int_0^t B_\varepsilon^{(i)}(s,y) \circ dW_s^i,$$

where:

$$\begin{aligned} A_\varepsilon(t,y) &= \int_{\mathbb{R}^d} u(t,x)\left\{b(t,x)\cdot D_x\left[\theta_\varepsilon(y-x)\right] + \operatorname{div} b(t,x)\theta_\varepsilon(y-x)\right\}dx, \\ B_\varepsilon^{(i)}(t,y) &= \int_{\mathbb{R}^d} u(t,x)D_i\left[\theta_\varepsilon(y-x)\right]dx. \end{aligned}$$

The proof will be divided in two steps. First we will prove:

$$u_\varepsilon(t,\phi_t(x)) = -\int_0^t \mathcal{R}_\varepsilon[b_s,u_s](\phi_s(x))ds.$$

Then we will prove:

$$s \to \int_{\mathbb{R}^d} \mathcal{R}_\varepsilon[b_s,u_s](\phi_s(x))\rho(x)dx,$$

satisfies the assumptions of Lebesgue dominated convergence theorem $\mathbb{P}$-almost surely on $[0,T]$ and converges to zero. The convergence to zero works as Chapter 2 in [3].

Let us start with the first step. Again, denote $\phi_t(x)$ to be the stochastic flow associated to (2). Applying Kunita-Itô-Wentzel formula (see [2]) one has:

$$\begin{aligned} du^\varepsilon\left(t,\phi_t(x)\right) =& A_\varepsilon\left(t,\phi_t(x)\right)dt + \sum_{i=1}^d B_\varepsilon^{(i)}\left(t,\phi_t(x)\right)\circ dW_t^i \\ &+ \left(b\cdot Du^\varepsilon\right)\left(t,\phi_t(x)\right)dt + \sum_{i=1}^d \left(D_i u^\varepsilon\right)\left(t,\phi_t(x)\right)\circ dW_t^i. \end{aligned}$$

Noticing:

$$\left(D_i u^\varepsilon\right)(t,y) = -\int_{\mathbb{R}^d} u(t,x)D_i\left[\vartheta_\varepsilon(y-x)\right]dx,$$

the first step is concluded. □

5. Appendix

Lemma 3. *(Generalized Itô's formula) Let $U : [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$ be a function in $\mathbb{L}^\infty(0,\infty; C_b^{2+\alpha}(\mathbb{R}^d))$ such that:*

$$U(t,x) - U(s,x) = \int_s^t V(r,x)dr,$$

for every $t \geq s \geq 0$, $x \in \mathbb{R}^d$, with $V \in \mathbb{L}^\infty(0,\infty; C_b^\alpha(\mathbb{R}^d))$. Let $(X_t)_{t\geq 0}$ be a continuous adapted process of the form:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s,$$

where b and σ are progressively measurable, b integrable and σ square integrable in t with probability 1. Then:

$$U(t,X_t) = U(0,x) + \int_0^t \left(V + b_s \cdot \nabla U + \frac{1}{2} Tr\left(\sigma_s \sigma_s^T \nabla^2 U\right)\right)(s,X_s)ds + \int_0^t \nabla U(s,X_s)\sigma_s dW_s.$$

Proof. Set $U_\varepsilon(t,x) = \varepsilon^{-1}\int_t^{t+\varepsilon} U(s,x)ds$ and $V_\varepsilon(t,x) = \varepsilon^{-1}\int_t^{t+\varepsilon} V(s,x)ds$. One sees:

$$\partial_t U_\varepsilon = V_\varepsilon.$$

Therefore U_ε satisfies the assumptions of the classical Itô formula (because it is twice derivable in time, and twice in space). Notice now that:

$$|V_\varepsilon|, |U_\varepsilon|, \|\nabla U_\varepsilon\|, \left\|\nabla^2 U_\varepsilon\right\|$$

are all uniformly bounded in (t,x,ε). Finally $\int_0^t \nabla U_\varepsilon(s,X_s)\sigma_s dW_s$ converges in probability to $\int_0^t \nabla U(s,X_s)\sigma_s dW_s$ because $\int_0^t \left\|\sigma_s^T\left(\nabla U_\varepsilon(s,X_s) - \nabla U(s,X_s)\right)\right\|^2 ds$ converges in probability to zero. □

References

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