STOCHASTIC CALCULUS FOR FINANCE @ IBS - CHEAT SHEET

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Abstract. This document is a quick-reference "cheat sheet" for the Stochastic Calculus for Finance course within the Risk Management module that I assist in teaching @ IBS. The few exercises you will find here will be classified with the following labels: easy, medium, hard, very-hard.

Contents

1. Random variables and σ-algebras

Definition 1. *Given a set Ω, one defines $\mathcal{F} \subset \mathcal{P}(\Omega)$ σ-algebra if:*

(1) $\emptyset \in \mathcal{F}$,
(2) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$,
(3) $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{F} \implies \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

Definition 2. *A function $X : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ is called measurable if:*

$$X^{-1}(A) \in \mathcal{F}, \quad \text{for all } A \in \mathcal{E}.$$

Definition 3. *The σ-algebra generated by X is defined as:*

$$\sigma\left(\left\{X^{-1}(A) \ \text{ for } A \in \mathcal{E}\right\}\right)$$

Remark 1. *Someone asked me about the following property during the class. Here is the answer: Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $X, Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and $f : \mathbb{R} \to \mathbb{R}$. Is it always true that if $Y = f(X)$, with f non-injective, then one can conclude $\sigma(X) \not\subset \sigma(Y)$? The answer is subtle. A first trivial remark is that if f is only injective outside of the image of X then the answer is of course no, it is not always true. But something more can be said. If there exist two disjoint sets $A, B \in \mathcal{B}(\mathbb{R})$ such that:*

$$f(A) = f(B) \in \mathcal{B}(\mathbb{R}),$$

and:

$$\mathbb{P}(X \in A) > 0, \ \mathbb{P}(X \in B) > 0,$$

then yes, $\sigma(X) \not\subset \sigma(Y)$, not even almost surely. This is because if it was $\sigma(X) \subset \sigma(Y)$, there there would exist $g : \mathbb{R} \to \mathbb{R}$ such that:

$$X = g(Y), \ \text{a.s.}$$

and therefore $X = g \circ f(X)$ a.s. that implies that, denoting with $\mathcal{L}_X$ the pushforward law of X, $Id = g \circ f$ $\mathcal{L}_X$-a.s. But this is a contradiction because $\mathcal{L}_X(A) > 0$, $\mathcal{L}_X(B) > 0$, and $A \cap B = \emptyset$, but $g \circ f(A) = g \circ f(B)$ and $Id(A) = A \neq B = Id(B)$.
But you can construct non-injective functions f, random variables X, Y, such that $Y = f(X)$ and $\sigma(X) \subset \sigma(Y)$ almost surely.

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2. Finite dimensional asset markets

2.1. Introduction. In a finite dimensional asset market, we model the payoff of the market with a matrix:

$$A \in \mathbb{R}^{m \times n}$$

where n is the number of different assets in the market, and m refers to the number of different states that the market can have. The prices are usually stored in:

$$S \in \mathbb{R}^n.$$

When given a new asset with payoff $b \in \mathbb{R}^m$, and you are asked if this can be perfectly hedged, mathematically you just need to verify if:

$$b \in \operatorname{Span}(A),$$

where $\operatorname{Span}(A) := \operatorname{Span}(A_1, \dots, A_n)$, and where $A_1, \dots, A_n$ are the columns of A.

2.2. Definition of arbitrage.

(1) **First type:** $\exists x$ such that $S \cdot x \le 0$, $Ax \ge 0$ and $Ax \ne 0$.
(2) **Second type:** $\exists x$ such that $S \cdot x < 0$, $Ax = 0$.

Theorem 1. *there is no type 2 arbitrage if and only if $\exists \psi$ such that $S = A^T \psi$.*

Theorem 2. *there is no type 1 or 2 if and only if $\exists \psi >> 0$ such that $S = A^T \psi$.*

Remark 2. *let us define $R_f := \frac{1}{\mathbb{I} \cdot \psi}$ where:*

$$\mathbb{I} := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

and define $q := R_f \psi$. Notice that $q \cdot \mathbb{I} = 1$.
The fair price of the new asset b is then:

$$\psi \cdot b = \frac{1}{R_f} q \cdot b = \frac{1}{R_f} \mathbb{E}_q [b]$$

The idea is: the price is fair when:

$$\mathbb{E}_q [\mathit{price} \cdot R_f - \mathit{payoff}] = 0$$

being q the risk-neutral probability.

3. Infinite dimensional asset markets

3.1. Random variables and conditional expectation.

Definition 4. *given $(E, \mathcal{E})$ and $(F, \mathcal{F})$ two measurable spaces, we say that $X : E \to F$ is measurable if $X^{-1}(A) \in \mathcal{E}$ for all $A \in \mathcal{F}$.*

Theorem 3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Given $\mathcal{G} \subset \mathcal{F}$ another σ-algebra contained in $\mathcal{F}$, and given $X : \Omega \to \mathbb{R}$ a random variable with $\mathbb{E}[|X|] < +\infty$ there exists a unique random variable Y such that:*

$$\mathbb{E}[XZ] = \mathbb{E}[YZ],$$

for all Z that is $\mathcal{G}$-measurable.

Definition 5. *we call the random variable Y from the previous theorem $\mathbb{E}[X|\mathcal{G}]$.*

Proposition 1. *The following properties hold:*

*(1) **Tower property:** if $\mathcal{G} \subset \mathcal{H}$ then:*

$$\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{G}].$$

4. Additional exercises

Exercise 1. *(medium) Is it true that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras then $\mathcal{F}_1 \cup \mathcal{F}_2$ is σ-algebra? If yes, prove it. If no, exhibit a counterexample.*

Exercise 2. *(very-hard) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\alpha \in (0, 1)$. Is it always true that you can find $A \in \mathcal{F}$ such that $\mathbb{P}(A) = \alpha$? Characterize when the answer is yes, and when it is no.*

References